There is no royal road to geometry (Euclid)

Yakov Eliashberg *is one of the leading mathematicians of our time. For more than thirty years he has helped to shape and research a field of mathematics known as symplectic geometry, solving many of its most important problems and finding new and surprising results. He has further developed the techniques he used in contact geometry, a twin theory to symplectic geometry. Both theories are closely related to current developments in modern physics.*

The universe is geometric. Albert Einstein presented this revolutionary idea one hundred years ago, in his general theory of relativity. In this theory, gravity is no longer described as a Newtonian attraction force that acts on different masses, instead it is the curvature of the geometry of space and time. The foundations of this geometry were laid in the mid-1800s by the German mathematician Bernhard Riemann. His aim was to develop geometry that described the very large and the very small – which was exactly what then happened.

However, geometry is much older than this. It is one of the oldest sciences, with roots in ancient Egypt and Babylonia around 5,000 years ago. The word geometry originates from the Greek: *geo*, which means earth, and *metria* – measure. Indeed, geometry was originally about practical needs such as measuring and distributing land, constructing buildings or calculating astronomical numbers. It dealt with surfaces, figures and shapes, with quadrants and cubes, circles and spheres. With parallel lines that never meet and triangles where the angles add up to 180 degrees. The Greek Euclid collected and formulated around 300 B.C. all the Antique geometric knowledge in one work, *Elementa*, and his Euclidean geometry is still taught in schools.

Geometry can come in many different forms; simply drawing a triangle on the curved surface of the Earth is sufficient to realise that there must be other types of geometry. For example, in a triangle with one corner on the North Pole and two on the equator, all the angles can be 90 degrees and then add up to 270 degrees.

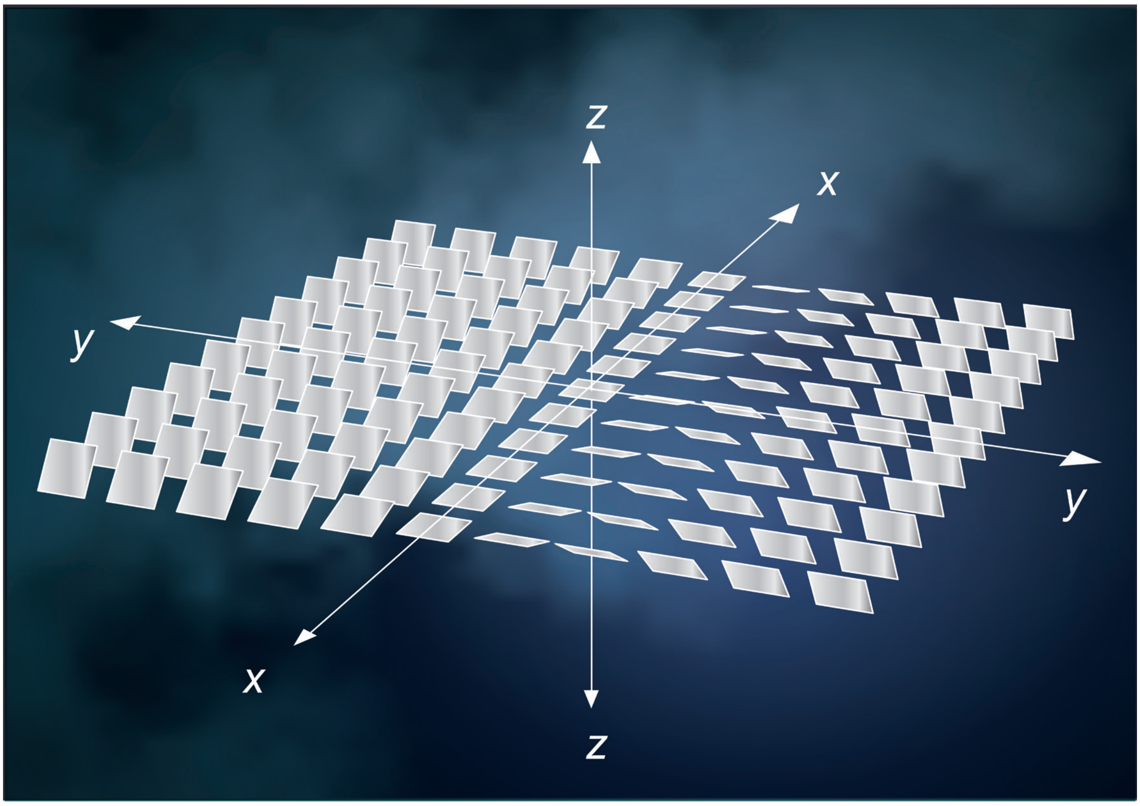
Bernhard Riemann did not just investigate curved two-dimensional surfaces, but broadened the concept to multi-dimensional spaces, called *manifolds*, and presented his discoveries in a famous lecture in Göttingen in 1854. The Riemannian geometry of curved space was an invaluable tool for Albert Einstein when he described, in his general theory of relativity, how empty space is curved by the mass of stars, galaxies and galaxy groups.

In addition to Euclidean and Riemannian geometries, there is also a lesser known type of geometry that has even deeper roots in physics: symplectic geometry. Somewhat simplified, Riemannian geometry can be said to be the expansion of Euclidean geometry to curved space with several dimensions. Symplectic geometry can be described in the same way – as a curvilinear expansion of the well-known Euclidean geometry.

However, there are significant differences between Riemannian and symplectic geometry; these have not yet been completely investigated, even though the roots of symplectic geometry go back several hundred years. Symplectic geometry was first used to study the classical mechanics that originated in Isaac Newton's laws of motion in the late 1600s. It thus comprises the very foundation of classical physics.

A classical mechanical system could be a planet orbiting the sun, an electron moving in an electromagnetic field, a swinging pendulum or a falling apple. In the 19th century, developments in classical physics led to the simplification of the often complicated calculations that used Newton's differential equations. Instead, methods were introduced to describe and understand motion in terms of symplectic geometry. Physics became geometry.

For example, symplectic geometry describes the geometry of a space that consists of the position and momentum values of a mechanical system, the phase space. For a moving object, its trajectory is determined each moment by its position and velocity, a pair of parameters. Together, they determine a surface element that is the basic structure of symplectic geometry. The geometry describes the directions in which the system can develop; it describes movement.



Contact structure.

In the 1800s, mathematicians demonstrated that these surface elements are preserved, i.e. remain constant over time, which has become an important characteristic of symplectic geometry. The surface element has also received a quantum physics interpretation in Heisenberg's uncertainty principle, which says that it is not possible to exactly measure the position and the velocity of a particle simultaneously. This means you can think of a symplectic surface element as a measure of the combined values of position and momentum.

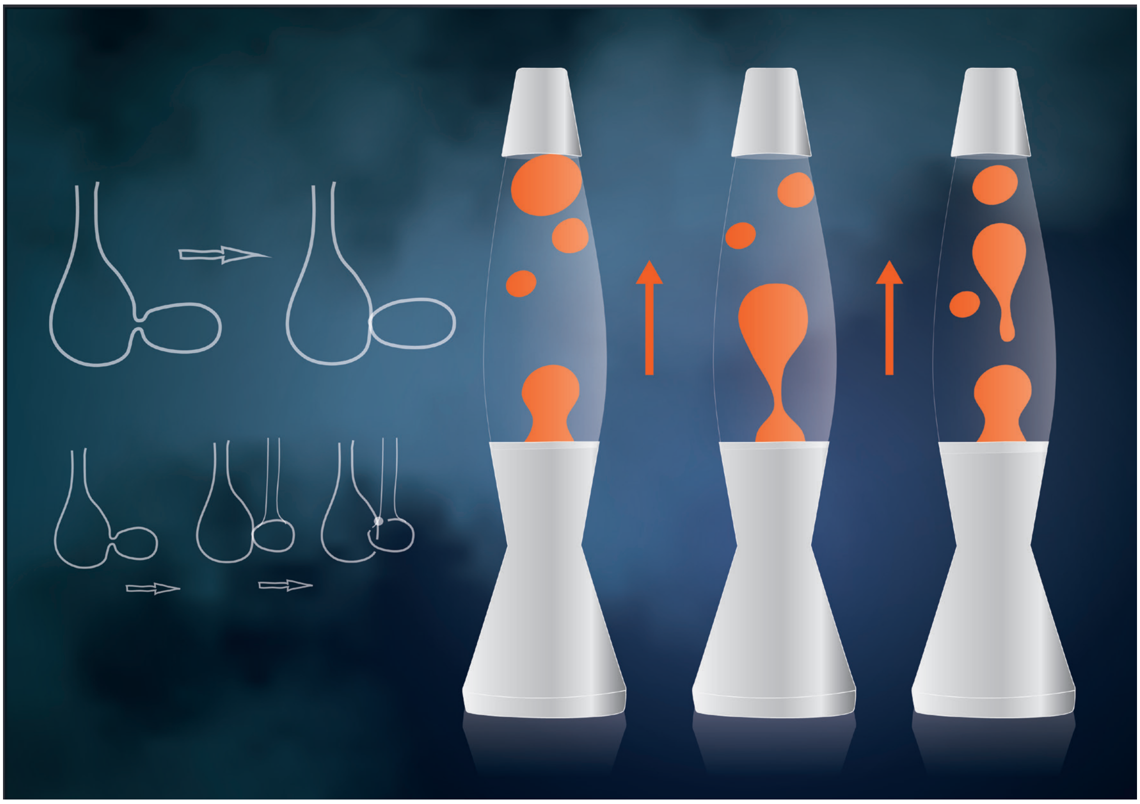
These two values lead to a plane, two-dimensional geometry. However, it can be generalised to geometries in four, six or more even dimensions, which have become interesting objects of study for both mathematicians and physicists in recent decades.

The modern development of symplectic geometry originates in the works of the Russian mathematician Vladimir Arnold in the late 1970s. He was awarded the very first Crafoord Prize for Mathematics in 1982.

Arnold formulated a number of central problems for which solutions were needed. Yakov Eliashberg was among those who were inspired; it is not easy to do his efforts complete justice in such a short text. He has been a leading figure in the field since the 1980s, and his ground-breaking research has both expanded and deepened symplectic geometry and its related areas, some of which he developed himself.

One of his first and perhaps most surprising results was the discovery that there are regions where symplectic geometry is rigid and other regions where it is completely flexible. Flexibility is studied in a branch of mathematics called topology. Here, geometric objects can be arbitrarily stretched, twisted, and bent without losing their properties. In contrast, deformations of rigid geometric objects are much more constrained and are determined by more refined properties than merely their topology.

In symplectic geometry, rigidity and flexibility live side by side and one of the challenges was to find the decisive properties of geometric objects that determine whether they are flexible or rigid. Time and again, the work of Eliashberg has showed that the borderline between the two regions is not where it first appears to be.



Holomorphic curves.

One of the characteristic properties of symplectic geometry is that on a small scale, i.e. locally, all symplectic spaces look the same despite being different globally. This is in contrast to ordinary geometry, where there are important local differences: if you have a ball, all you have to do is to draw a small triangle on its surface to discover that it is curved. This is not the case for symplectic geometry.

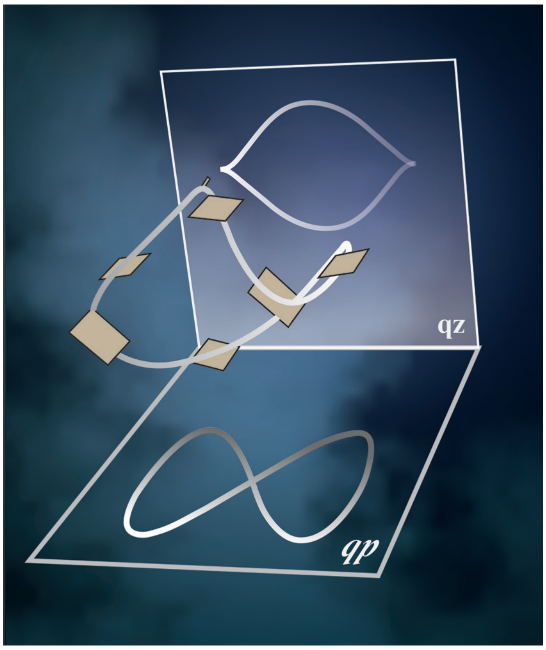
Yakov Eliashberg has identified the smallest building blocks, the atoms of flexibility. The presence of such an atom guarantees that everything is the same, even on a larger scale, and that you are in the flexible region. Eliashberg's flexibility theory says that a rigid region can be transformed into a flexible one if you introduce a single such tiny building block. For example, a construction that is as stiff and stable as the Eiffel Tower would lose its rigidity if a tiny flexible building block was added; the tower would become limp and collapse.

However, if there is no such flexible building block, you are in our rigid world in which things retain their shape. This world is built from tiny strings according to string theory, which tries to unify the two most successful physics theories of the twentieth century – quantum mechanics and the general theory of relativity. String theory has long had an intensive exchange of ideas with symplectic geometry, where in particular symplectic rigidity plays an important role.

A now classic example of symplectic rigidity is the *non-squeezing* theorem, which Vladimir Arnold also named the principle of the *symplectic camel* after a reformulated biblical quote: "It is easier for a rich man to enter the kingdom of heaven than it is for a symplectic camel to pass through the eye of a needle".

It was the Russian-French mathematician and Eliashberg's colleague Mikhail Gromov who demonstrated that symplectic geometry does not permit the circus trick of pulling a camel through a needle's eye. If the issue was just the volume, the camel could be stretched out into a long thin thread. But symplectic geometry does not allow this; rigidity is what applies here.

The answer to the camel problem was obtained using Gromov's theory of holomorphic curves, which was later related to both string theory and the quantum field theory of high energy physics. It also became the central tool in the new field of symplectic topology.



Legendrian unknot.

Eliashberg has transferred much of this mathematics to contact geometry, a sister theory to symplectic geometry which lives in odd dimensions. An odd dimension can be achieved when the total energy in a mechanical system is constant (according to the principle that energy in a closed system can neither be created nor destroyed). This means that the system cannot move freely in its two degrees of freedom, so one dimension can be removed.

If, for example, this constant energy determines the distance from the centre of a plane, contact geometry will apply to the one-dimensional circle around the centre, while symplectic geometry applies in the two dimensional plane surrounding it. The two geometries are thus very closely related, and contact geometry in one or more odd dimensions has been another of Yakov Eliashberg's specialist areas.
In the intersection between contact geometry and mathematical knot theory there are Legendre knots. These are knots that must follow special limitations dictated by contact

geometry. Using the same type of technique as for the camel, Eliashberg demonstrated that it is not always possible to transform two Legendre knots into each other, even if there are no purely topological barriers. The knots are rigid.

The large mathematical machinery developed by Eliashberg for studies of rigidity led to a new field – symplectic field theory – which, in turn, became a source of numerous new insights, discoveries and links to other areas.

Time and again, Yakov Eliashberg has found new areas and problems that are of particular interest to explore. But where the boundary is between the flexible and the rigid regions, and how it can be described mathematically, is still a question that is awaiting an answer.

THE LAUREATE

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LINKS AND FURTHER READING

More information about this year's prize is available at www.crafoordprize.se and the Royal Swedish Academy of Sciences' website, *http://kva.se/crafoordprize*

Lectures by Yakov Eliashberg (video)

http://scgp.stonybrook.edu/video_portal/results.php?profile_id=985

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